

All homogeneous pure radiation spacetimes satisfy the Einstein-Maxwell equations

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Abstract

It is shown that all homogeneous pure radiation solutions to the Einstein equations admit electromagnetic sources. This corrects an error in the literature.

A spacetime (M, g) is said to be a homogeneous pure radiation solution of the Einstein equations if it admits a transitive group of isometries and if there exists a function Φ and 1-form k such that the Einstein tensor G is of the form

$$G_{ab} = \Phi^2 k_a k_b, \quad \text{where} \quad g^{ab} k_a k_b = 0. \quad (1)$$

A classification of homogeneous pure radiation solutions is given in Ref. [1] based upon results of Refs. [2] and [3]. Such solutions are all pp waves, and they are either plane waves or they are diffeomorphic to

$$g = dx \otimes dx + dy \otimes dy + du \odot dv - 2e^{2\rho x} du \otimes du, \quad \rho = \text{const.} \quad (2)$$

The plane waves are known to arise from electromagnetic sources. It is asserted in Refs. [1, 3], based upon results of Ref. [4], that the metric (2) does not arise from an electromagnetic source. However, the metric (2) *does*

arise from an electromagnetic source. In fact, infinitely many electromagnetic fields can serve as the source for the spacetime defined by (2).

For any function $f(u)$ the null two form

$$F = 2\rho e^{\rho x} \left[\cos(\rho y + f(u)) du \wedge dx - \sin(\rho y + f(u)) du \wedge dy \right], \quad (3)$$

its Hodge dual \tilde{F} , and the metric (2) satisfy the source-free Maxwell equations $dF = 0 = d\tilde{F}$ and the Einstein equations

$$G_{ab} = \frac{1}{2} (F_{ac} F_b{}^c + \tilde{F}_{ac} \tilde{F}_b{}^c) = 4\rho^2 e^{2\rho x} \nabla_a u \nabla_b u. \quad (4)$$

The electromagnetic field given in (3) is “non-inheriting”, that is, it is not invariant under the isometry group of (M, g) . For example, F is not invariant under translations in y for any choice of $f(u)$.

Thus, granted Theorem 12.6 of [1], all homogeneous pure radiation solutions of the Einstein equations arise from electromagnetic sources.

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References

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